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A classification of semisimple symmetric pairs and their restricted root system

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1. Introduction

Many results about the structure theory of real reductive Lie algebras can be proved in the more general setting of pairs (G,θ), where G is a reductive algebraic group defined over an algebraicly closed field F of characteristic $\neq 2$ and θ an involutorial automorphism of G.

The idea behind this is the following. Let $\mathfrak{g}$ denote the Lie algebra of G and denote the automorphism of $\mathfrak{g}$ induced by θ also by θ. If $F = \mathbb{C}$, then there exists a compact real form $\mathfrak{u}$ of $\mathfrak{g}$ with conjugation τ, which is θ-stable. Now the fixed point set $\mathfrak{g}_{\theta\tau} = \{X \in \mathfrak{g} \mid \theta\tau(X) = X\}$ of $\theta\tau$ is a real form of $\mathfrak{g}$ and $\theta|_{\mathfrak{g}_{\theta\tau}}$ is a Cartan involution of $\mathfrak{g}_{\theta\tau}$. Conversely by complexifying a Cartan involution of a real form of $\mathfrak{g}$ we obtain an involutorial automorphism of $\mathfrak{g}$.

It was already shown by Cartan [6], that this gives a bijective correspondence between the set of isomorphism classes of involutorial automorphisms of $\mathfrak{g}$ and the set of isomorphism classes of real forms of $\mathfrak{g}$. Using this translation several results about (G,θ) transfer to $\mathfrak{g}_{\theta\tau}$

The aim of this contribution is to collect a number of the known results about these pairs (G,θ), and to give a sketch how this machinery can be used to obtain a fairly easy classification of locally affine symmetric spaces with their restricted root system (see section 3).

2. Review of results about pairs (G,θ)

We use as our basic references for algebraic groups the books of Humphreys [14] and Springer [25] and we shall follow their notations and terminology.

The basic data about the pairs (G,θ) were developed by Vust [30] and Richardson [20]. In this section we will recollect some of their results. First some notations. Let G denote a reductive algebraic group, $\mathfrak{g}$ its Lie algebra and $\theta \in \mathrm{Aut}(G)$ an involution of G. We denote by $K = \{g \in G \mid \theta(g) = g\}$ the group of fixed points of θ. It is a closed reductive subgroup of G (see Vust [30]). Let T be a torus of G. We set $W(T) = N_G(T)/Z_G(T)$ for the Weyl group of G with respect to T. It acts by conjugation on T, on $X^*(T)$, the group of characters of T and on $X_*(T)$, the group of one parameter subgroups of T. If $\alpha \in X^*(T)$, then $\mathfrak{g}_\alpha$ denotes the weight space for the character α on the Lie algebra $\mathfrak{g}$, where T acts on $\mathfrak{g}$ by the adjoint representation. Write $\Phi(T)$ for the set of roots of G with respect to T.

If T is a θ-stable torus of G, we set $T_\theta^+ = (T \cap K)^\circ$ and $T_\theta^- = \{x \in T \mid \theta(x) = x^{-1}\}^\circ$. One easily verifies that the product map

$$\mu: T_\theta^+ \times T_\theta^- \to T, \quad \mu(t_1,t_2) = t_1 t_2$$

is a separable isogeny. So in particular $T = T_\theta^+ . T_\theta^-$ and $T_\theta^+ \cap T_\theta^-$ is a finite group. (In fact it is an elementary abelian 2-group.) If T is a θ-stable torus of G, then the automorphisms of $\Phi(T)$ and W(T) , induced by θ, will also be denoted by θ.

2.1. Twisted action

The twisted action of G on G is denoted by:

$$(g,x) \to g^*x = gx\theta(g)^{-1}$$

If $x \in G$, we denote by $G^*x = \{gx\theta(g)^{-1} | g \in G\}$. Let $S = G^*e = \{g\theta(g)^{-1} | g \in G\}$. We then have:

2.2. Proposition. *S is a closed subvariety of G on which G acts transitively by* $g^*x = gx\theta(g)^{-1}$. *The map* $g \to g^*e = g\theta(g)^{-1}$ *induces an isomorphism of affine G-varieties* $\sigma: G/K \to S$.

This is proved in Richardson [20].

For $F = \mathbb{C}$, the space G/K is the complexification of a space $G(\mathbb{R})/K(\mathbb{R})$ with a $G(\mathbb{R})$-invariant Riemannian structure. Here $G(\mathbb{R})$ (resp. $K(\mathbb{R})$) denotes the set of $\mathbb{R}$-rational points of G (resp. K.)

Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ resp. $\mathfrak{p}$ are the $+1$ resp. -1-eigenspace of θ, it follows that KS is an open dense subset of G, but in general $G \neq KS$ and also $K \cap S$ need not be reduced to $\{e\}$. This differs from the Cartan decomposition in the case of a real semisimple Lie group, which possesses these properties.

Similarly we have no Iwasawa decomposition, but there is a natural notion of minimal parabolic subgroup for these pairs (G,θ), which coincides for $F = \mathbb{C}$ with the complexification of a minimal parabolic subgroup of the corresponding real Lie group.

2.3. θ-split parabolic subgroups and tori

A torus A of G is called *θ-split* if $\theta(x) = x^{-1}$ for all $x \in A$. A parabolic subgroup P of G is called *θ-split* if P and $\theta(P)$ are opposite (i.e. if $P \cap \theta(P)$ is a common Levi factor of both P and $\theta(P)$).

The following properties about θ-split tori and parabolic subgroups are due to Vust [30].

2.4. Proposition. (*i*) *If* $\theta \neq \mathrm{id}$, *then G contains a non-trivial θ-split torus;*

(*ii*) *if A is a maximal θ-split torus of G, then there exists a minimal θ-split parabolic subgroup P of G such that* $P \cap \theta(P) = Z_G(A)$;

(*iii*) K° *acts transitively on the set of minimal θ-split parabolic subgroups of G and also on the set of maximal θ-split tori of G.*

We also have a notion of restricted root system for a pair (G,θ):

2.5. Proposition. *Let A be a maximal θ-split torus of G and* $E_0 \subset X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}$ *be the vector space spanned by* $\Phi(A)$. *Then* $\Phi(A)$ *is a root system in* E_0 *with Weyl group* $W(A)$. *Moreover every* $w \in W(A)$ *has a representative in* $N_{K^\circ}(A)$.

This is proved in Richardson [20,§4]

Since all maximal θ-split tori are conjugate under K°, the root system $\Phi(A)$ is uniquely determined by (G,θ), up to isomorphism. If $F = \mathbb{C}$, then $\Phi(A)$ coincides with the restricted root system of the Riemannian symmetric space $G(\mathbb{R})/K(\mathbb{R})$, where $G(\mathbb{R})$ and $K(\mathbb{R})$ are as in 2.2 (see Helgason [9]).

For G simple the pairs (G,θ) together with this restricted root system are classified in Helminck [11]. It is shown that this list is the same as the list of real forms of a complex simple Lie algebra. (See for example Sugiura [22,appendix]).

2.6. θ-stable maximal tori

In [29] Vogan showed that for $F = \mathbb{C}$, there is a bijection of the set of K^o-conjugacy classes of θ-stable maximal tori of G onto the set of $\mathrm{Int}(\mathfrak{g}_{\theta_r})$-conjugacy classes of Cartan subalgebras of $\mathfrak{g}_{\theta_r}$, where $\mathfrak{g}_{\theta_r}$ is a real Lie algebra corresponding to (G,θ) as in 1. As one can expect the same holds for all other characteristics. In fact these orbits can be characterized by W(T)-conjugacy classes of involutions of a certain maximal torus T of G. To be more exact:

2.7. Theorem. *Let* A *be a maximal θ-split torus of* G *and* $T \supset A$ *a maximal torus of* G*. There is a bijective correspondence between the set of* K^o*-conjugacy classes of θ-stable maximal tori of* G *and the* W(T)*-conjugacy classes* $\mathcal{W}$ *of involutions in* W(T) *with the property that* $T_w^- \subset A$ *for some representative* w *of* $\mathcal{W}$*.*

These results can be found in Helminck [10], where also the diagram of K^o-conjugacy classes of θ-stable maximal tori of G is given. This work includes also the results of Schmidt [23] and Hirai [13] on Cartan subalgebras.

2.8. K-orbits on the flag variety

The K^o-conjugacy classes of θ-stable maximal tori can be used to describe the set of double cosets $B\backslash G/K$, where B is a θ-stable Borel subgroup of G. The existance of this θ-stable Borel subgroup of G is a particular case of a more general result of Steinberg [27,p. 51]. There exists also a maximal torus $T \subset B$, which is θ-stable. In the corresponding real situation T corresponds to a fundamental Cartan subalgebra of the real Lie algebra $\mathfrak{g}_{\theta_r}$ (see Warner [31]). Such a pair (B,T) can also be characterized by the conditions that $B \cap K^o$ is a Borel subgroup of K^o and $T_1 = T \cap K^o$ is a maximal torus of K^o.
This pair (B,T) is fundamental in the following results due to Springer [26]:

2.9. Theorem. *Let* B *be a θ-stable Borel subgroup of* G *and* $T \subset B$ *a maximal torus of* G *with* $\theta(T) = T$*. There is a bijection of the set* $B\backslash G/K$ *of double cosets onto the* K*-orbits in the set of pairs* (T_1, B_1)*, where* T_1 *is a θ-stable maximal torus of* G *and* B_1 *a Borel subgroup containing* T_1*.*

This result follows from the following description of the orbits. Let (B,T) be as above, write $N = N_G(T)$ and for $g \in G$ put $\tau g = g(\theta g)^{-1}$. The group $T \times K$ acts on $\tau^{-1}N$ by $(t,k)g = tgk^{-1}$. Let V be the set of these orbits. We then have:

2.10. Theorem. (*i*) V *is finite*
(*ii*) *G is the disjoint union of the double cosets* BvK ($v \in V$).

Now the bijection in 2.9 is given by the map

$$BvK \rightarrow (\dot{v}^{-1}T\dot{v}, \dot{v}^{-1}B\dot{v}),$$

where $\dot{v}$ is a representative of v in $\tau^{-1}N$.

Over $\mathbb{C}$ (2.9) was first established by Rossmann [21] and Matsuki [16]. A generalization of the results in [16] to general fields of characteristic not 2 can be found in [12]. A description of the orbit closures is given in [25].

These K-orbits in $B\backslash G$ are of importance in the representation theory of real Lie groups (see Vogan [28,p.382]).

2.11. Remark. As a last result we mention that for G semi-simple and adjoint, there exists a very nice "compactification" of the affine variety G/K in the context of algebraic geometry. (see de Concini and Procesi [7]). A similar compactification of the real symmetric spaces was given by Oshima and Sekiguchi [18,§2].

2.12. Example. Let $G = SL_n(F)$ and for $g \in G$ let $\theta(g) = ({}^t g)^{-1}$. Then $K = SO_n(F)$ and $S = \{g^t g \mid g \in SL_n(F)\}$ is the set of symmetric matrices in $SL_n(F)$.
The diagram of K-conjugacy classes of θ-stable maximal tori is totally ordered, using as invariant the dimension of T_θ^-. In this case $A = \{\text{diagonal matrices in } SL_n(F)\}$ is a maximal θ-split torus of G, which is also a maximal torus. (We note that for every connected reductive group G there exists exactly one isomorphism class of involutorial automorphisms θ of G, such that G contains a maximal torus, which is θ-split. In the corresponding real situation this corresponds with the case that $\mathfrak{g}_{\theta\tau}$ is a normal real form (see Helgason [9]).)
If B is a θ-stable Borel subgroup of $SL_n(F)$, then the elements of $B \backslash G$ can be intrepretated as complete flags in a projective space $\mathbb{P}^{n-1}$ and the elements of the symmetric space G/K as the non-degenerate quadrics in $\mathbb{P}^{n-1}$. So the double cosets BgK describe the various possible positions which such a flag can have with respect to a given quadric.

3. Pairs of commuting involutorial automorphisms

In this section we shall give a sketch of a classification of pairs of commuting involutorial automorphisms of G and show that these results also give a classification of locally affine symmetric spaces with their fine stucture. These spaces are defined as follows:

3.1. Definition Let $\mathfrak{g}_o$ denote a real semisimple Lie algebra and let $\sigma \in \mathrm{Aut}(\mathfrak{g}_o)$ be an involution (i.e. $\sigma^2 = \mathrm{id}$) and let $\mathfrak{h} = (\mathfrak{g}_o)^o_\sigma = \{X \in \mathfrak{g}_o \mid \sigma(X) = X\}$ denote the set of fixed points of σ. The pair $(\mathfrak{g}_o,\mathfrak{h}) = (\mathfrak{g}_o,\sigma)$ is called a *locally affine symmetric pair* and the corresponding symmetric space $\mathfrak{g}_o/\mathfrak{h}$ a *locally affine symmetric space.*

To get the correspondence with the pairs of commuting involutions we still need a second involution. For this we take the Cartan involution of $\mathfrak{g}_o$, which determines the real form up to isomorphism (see section 1). It is not hard to prove that there exists a Cartan involution, which commutes with σ:

3.2. Proposition. *Let $(\mathfrak{g}_o,\sigma)$ be a locally affine symmetric pair. Then there exists a Cartan involution θ of $\mathfrak{g}_o$, such that $\sigma\theta = \theta\sigma$.*

For a proof see Berger [2].

If $\mathfrak{g} = \mathfrak{g}_o \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of $\mathfrak{g}_o$, then complexifying both involutions to $\mathfrak{g}$, we obtain a pair of commuting involutions (σ,θ) of $\mathfrak{g}$.

On the other hand starting with two commuting involutions $\sigma,\theta \in \mathrm{Aut}(\mathfrak{g})$ we get two locally affine symmetric pairs as follows. Embed the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by σ and θ in a maximal compact subgroup U of $\mathrm{Aut}(\mathfrak{g})$. Its Lie algebra $\mathfrak{u}$ is a compact real form of $\mathfrak{g}$, which is σ- and θ-stable. Let τ be its conjugation. Then both $\sigma\tau$ and $\theta\tau$ are conjugations of $\mathfrak{g}$, which define real forms

$$\mathfrak{g}_{\theta\tau} = \{X \in \mathfrak{g} \mid \theta\tau(X) = X\} \text{ and } \mathfrak{g}_{\sigma\tau} = \{X \in \mathfrak{g} \mid \sigma\tau(X) = X\}.$$

Consequently $(\mathfrak{g}_{\theta\tau},\sigma|_{\mathfrak{g}_{\theta\tau}})$ and $(\mathfrak{g}_{\sigma\tau},\theta|_{\mathfrak{g}_{\sigma\tau}})$ are locally semisimple symmetric pairs, which are called dual. Since we get two locally affine symmetric pairs we need to consider *ordered pairs of* *commuting involutions* and we let (σ,θ) correspond to the first pair and (θ,σ) to the second. It is now not hard to show that all this is independent of the choice of τ (and also of the Cartan involution commuting with σ), so that we obtain the following bijective correspondence:

$$\left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{locally affine symmetric pairs}\end{array}\right\} \overset{1\text{-}1}{\longleftrightarrow} \left\{\begin{array}{c}\text{isomorphism classes of ordered pairs}\\ \text{of commuting involutions of}\\ \text{a complex semisimple Lie algebra}\end{array}\right\}$$

This last problem is in fact independent of the field of definition of the Lie algebra and in order to obtain also some results in characteristic p, the corresponding problem on the group is studied. It appears that the above isomorphism classes correspond bijectively to the isomorphism classes of ordered pairs of commuting involutions of G. Denote the set of these isomorphism classes by $\mathcal{C}$ and for a fixed maximal torus T of G write $\mathcal{C}(T,W)$ for the set of W(T)-conjugacy classes of ordered pairs of commuting involutions of $(X^*(T),\Phi(T))$.

The idea is now to restrict the problem of the classification of the classes in $\mathcal{C}$ by characterizing the pairs of commuting involutions of G on a fixed maximal torus T of G. In other words to define a map

$$\rho : \mathcal{C} \to \mathcal{C}(W,T)$$

and classify its image and the fibers. To construct such a map one could take in any class c of $\mathcal{C}$ a representative (σ,θ) such that T is σ- and θ-stable and take for $\rho(c)$ the W(T)-conjugacy class of $(\sigma|_T,\theta|_T)$. However then ρ is not well-defined. This definition leaves too much freedom for the choice of (σ,θ). Different representatives of the class c in $\mathcal{C}$, stabilizing T, can induce different classes in $\mathcal{C}(T,W)$. Hence we have to demand more properties of the representative. We shall describe now first the classification of single involutions.

3.3. Classification of single involutorial automorphisms

In this subsection we assume $\sigma = \theta$. Moreover let Z(G) denote the center of G, T a maximal torus of G and $\mathrm{Aut}(G,T) = \{\phi\in\mathrm{Aut}(G)|\phi(T) = T\}$. For any torus T_1 call the elements $\epsilon\in T_1$ such that $\epsilon^2\in Z(G)$, *quadratic elements* of T_1. From the isomorphism theorem (see Springer [25]) we know:

Theorem. *Let* $\theta_1,\theta_2\in\mathrm{Aut}(G,T)$ *be involutions such that* $\theta_1|_T = \theta_2|_T$.
Then $\theta_1 = \theta_2\mathrm{Int}(\epsilon)$ *for some* $\epsilon\in T$, $\epsilon^2\in Z(G)$.

From this result it follows that the W(T)-isomorphism class of $\theta_1|_T$ determines the isomorphism class of θ_1 up to a quadratic element in T. For the inverse implication we have to demand an extra property of the representative θ of the isomorphism class $c\in\mathcal{C}$. There are two possibilities:

(1) T_θ^+ *is a maximal torus of* G_θ,

(2) T_θ^- *is a maximal* θ*-split torus of* G.

In both cases all maximal tori with this property are conjugate under G_θ, so if we demand from the representative θ of c that T_θ^+ (or T_θ^-) is maximal, then ρ is well defined.

If G is simple, then we get in the first case $\theta|_T = \mathrm{id}$ or a diagram automorphism. So the classification is then a matter of determining the W(T)-conjugacy classes of quadratic elements of T. A classification of real semisimple Lie algebras along these lines has been carried out by Gantmacher [8] and later also by Murakami [17], using a description of the quadratic elements in a maximal torus of Borel and Siebenthal [4]. The disadvantage of this approach is that it does not give any information about the restricted root system of the corresponding symmetric space G/K. On the other hand in the second characterization one obtains $\Phi(T_\theta^-)$, which coincides with the natural root system of the symmetric space G/K. It appears moreover that in this case ρ is one to one:

3.4. Theorem. *Let $\theta_1,\theta_2 \in \mathrm{Aut}(G,T)$ be involutions such that $T_{\theta_i}^-$ is a maximal θ_i-split torus of G. Then $\theta_1|_T$ is W(T)-conjugate to $\theta_2|_T$ if and only if θ_1 is isomorphic to θ_2.*

For a proof see Helminck [11]. In the real case this result is due to Araki [1]. We can represent each class in $\rho(\mathcal{C})$ by a Satake-diagram, from which we obtain also the restricted root system with the multiplicities of the restricted roots.

3.5. Characterization of pairs of commuting involutions

To classify the locally semisimple symmetric pairs, Berger [2] made a choice analogous to Gantmacher (using Cartans classification of the real semisimple Lie algebras), but did not obtain any results concerning the "fine structure" of those spaces, like the restricted root system. Therefor we choose a method similar to that of Araki. First some notation. Let (σ,θ) be a pair of commuting involutorial automorphisms of G and T a maximal torus of G.

3.6. Definitions. (*i*) If T is (σ,θ)-stable then the torus $\{t \in T \mid \sigma(t) = \theta(t) = t^{-1}\}^{\circ}$ is called (σ,θ)-*split* and will be denoted by $T_{\sigma,\theta}^-$.
(*ii*) The pair (σ,θ) is called *normally related to* T if T is σ- and θ-stable and if $T_{\sigma,\theta}^-$, T_σ^-, T_θ^- are respectively maximal (σ,θ)-split, σ-split and θ-split.

As in the case of a single involution, $\Phi(T_{\sigma,\theta}^-)$ is the natural root system of the corresponding real symmetric pair. Every class in $\mathcal{C}$ contains a pair (σ,θ) which is normally related to T. Denoting the center of G by Z(G), we have furthermore:

3.7. Theorem. *Let (σ_1,θ_1) and (σ_2,θ_2) be pairs of commuting involutorial automorphisms of G, normally related to T. Then $(\sigma_1,\theta_1)|_T$ and $(\sigma_2,\theta_2)|_T$ are conjugate under W(T) if and only if there exists a $\epsilon \in T_{\sigma,\theta}^-$ with $\epsilon^2 \in Z(G)$ such that (σ_2,θ_2) is isomorphic to $(\sigma_1,\theta_1 \mathrm{Int}(\epsilon))$.*

This result implies that if we demand of the representative (σ,θ) of the class $c \in \mathcal{C}$, that (σ,θ) is normally related to T, then the mapping ρ is again well-defined. Denote the image of ρ by $\mathcal{Q}$ and the fibers above $\rho((\sigma,\theta))$ by $\mathcal{C}(\sigma,\theta)$. The ordered pairs of commuting involutions of $(X^*(T),\Phi(T))$, whose class in $\mathcal{C}(T,W)$ is contained in $\mathcal{Q}$, are called *admissible*.

3.8. Classification of pairs of commuting involutions

The W(T)-conjugacy classes of admissible pairs of commuting involutions of $(X^*(T),\Phi(T))$ can be described by a diagram, which can be obtained by gluing together two diagrams of admissible involutions under a combinatorical condition on the simple roots. From this one obtains all the fine structure of the corresponding locally semisimple symmetric pair (see Helminck [12]).

As to the classification of the classes in $\mathcal{C}(\sigma,\theta)$, it suffices to give a set of quadratic elements of a maximal (σ,θ)-split torus A of G, representing the classes in $\mathcal{C}(\sigma,\theta)$. These quadratic elements can be described by using a basis of $\Phi(A)$. Namely assume G is adjoint, $\bar{\Delta}$ a basis of $\Phi(A)$ and $\{\gamma_\lambda\}_{\lambda\in\bar{\Delta}}$ a dual basis in $X_*(A)$, the set of multiplicative one parameter subgroups of A. If $\epsilon_\lambda = \gamma_\lambda(-1)$, $\lambda \in \bar{\Delta}$, then $\epsilon_\lambda^2 = e$. There exists a subset $\Delta_1 \subset \bar{\Delta}$ such that $\{\epsilon_\lambda\}_{\lambda\in\Delta_1}$ is a set of quadratic elements representing the classes in $\mathcal{C}(\sigma,\theta)$. This subset Δ_1 of $\bar{\Delta}$ is determined by the action of the restricted Weyl group W(A) on the group of quadratic elements of A and the signatures of the roots in $\bar{\Delta}$. This completes the classification.

3.9. Remark. Finally we note that every class $\mathcal{C}(\sigma,\theta)$ contains a unique class of standard pairs. This seems to be the natural class to start with in the analysis on these symmetric spaces. For example, if $\sigma = \theta$, then the standard pair in $\mathcal{C}(\theta,\theta)$ is (θ,θ), which corresponds to a Riemannian symmetric space and the other pairs in $\mathcal{C}(\theta,\theta)$ correspond to the K_ϵ-spaces described in [18].

Recently Oshima-Sekiguchi [19] also determined the restricted root system of a locally semisimple symmetric pair, based on the classification of Berger.

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